Review of Math 150

1. Given that $\lim_{x\to -2} f(x)=4$, $\lim_{x\to -2} g(x)=-3$, and $\lim_{x\to -2} h(x)=0$ find

   (a) $\lim_{x\to -2}[f(x)-2g(x)]$

   (b) $\lim_{x\to -2}\dfrac{\sqrt{f(x)\cdot[g(x)]^2}}{h(x)+f(x)}$

2. If $\lim_{x\to a}[f(x)+g(x)]=2$ and $\lim_{x\to a}[f(x)-g(x)]=1$, find $\lim_{x\to a}[f(x)g(x)]$.

3. Use the squeeze theorem to evaluate $\lim_{x\to 0^+}\sqrt{x}\; e^{Sin(\pi/x)}$

4. Define $f(x)=\begin{cases} 1+x^4 & if \quad x \quad is \quad irrational \\ 1+2x^4 & if \quad x \quad is \quad rational \end{cases}$. Find $\lim_{x\to 0} f(x)$.

5. Use a δ-ϵ argument to prove that

   (a) $\lim_{x\to 5}(-2x+1)=-9$

   (b) $\lim_{x\to -2}(2x^2-5x+1)=19$

6. Let $f(x)=\begin{cases} \dfrac{x^2-x}{x^2-1} & if \quad x\neq 1 \\ 1 & if \quad x=1 \end{cases}$. Determine the points where *f* is discontinuous.

   Justify your answer.

7. Let $a>0$ be a positive real number. Define $f(x)=\begin{cases} x^2 & if \quad x<a \\ 3x & if \quad x\geq a \end{cases}$.

   What is the value of a if *f* is continuous on the entire real number line?

8. For what value of the constant c is the function *f* continuous on (-∞,∞)?

$$f(x)=\begin{cases} cx^2+2x & if \quad x<2 \\ x^3-cx & if \quad x\geq 2 \end{cases}$$

9. Find the values of *a* and b that make *f* continuous on (-∞,∞)?

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & if \quad x<2 \\ ax^2-bx+3 & if \quad 2 \le x < 3 \\ 2x-a+b & if \quad x \ge 3 \end{cases}$$

10. Use the Intermediate Value Theorem to show that the equation $x^4 + x - 3 = 0$ has a root in the interval (1, 2).

11. Use the Intermediate Value Theorem to show that the equation $\sin x = x^2 - x$ has a root in the interval (1, 2).

12. Suppose *f* is continuous on [1, 5] and the only solutions of the equation $f(x) = 6$ are x = 1 and x = 4. If $f(2) = 8$, explain why $f(3) > 6$.

13. A Tibetan monk leaves the monastery at 7:00 AM and takes his usual path to the top of the mountain, arriving at 7:00 PM. The following morning, he starts at 7:00 AM at the top and takes the same path back, arriving at the monastery at 7:00 PM. Use the Intermediate Value Theorem to show that there is a point on the path that the monk will cross at exactly the same time of day on both days.

14. Let $f : [0, 1] \to (0, 1)$ be a continuous function such that $0 < f(x) < 1$ for all x $\in [0, 1]$. Prove that the equation $f(c) = c^2$ has a solution for at least one c $\in$ [0, 1].

15. Evaluate $\lim\limits_{x\to-\infty} \dfrac{\sqrt{9x^6 - x}}{x^3+1}$

16. Evaluate $\lim\limits_{x\to\infty} \left(\sqrt{x^2+ax} - \sqrt{x^2+bx}\right)$

17. Evaluate $\lim\limits_{x\to\infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

18. Find the indicated limit if it exists. Do not use l'Hospital.

19. (a) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x}$ (b) $\lim\limits_{x\to 0} \dfrac{\sin 4x}{\sin 6x}$

(c) $\lim\limits_{t\to 0} \dfrac{\tan 6t}{\sin 2t}$ (d) $\lim\limits_{\theta\to 0} \dfrac{\tan a\theta}{\sin b\theta}$

(e) $\lim_{x\to 0}\frac{\sin 2x}{x^2}$ (f) $\lim_{h\to 0}\frac{\cos(h)-1}{\sin(h)}$

(g) $\lim_{x\to 0}\frac{x\sin x}{1-\cos x}$ (h) $\lim_{x\to 0}\frac{\sin 3x}{5x^3-4x}$

(i) $\lim_{x\to 0}\frac{\sin(3x)\sin(5x)}{x^2}$ (j) $\lim_{x\to 0}\frac{\sin(x^2)}{x}$

(k) $\lim_{x\to 1}\frac{\sin(x-1)}{x^2+x-2}$ (l) $\lim_{x\to\pi/4}\frac{1-\tan x}{\sin x-\cos x}$

(m) $\lim_{x\to\infty} x\sin(1/x)$ (n) $\lim_{x\to\infty} x\sin(5/x)$

20. Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

(a) $\lim_{h\to 0}\frac{(1+h)^{10}-1}{h}$ (b) $\lim_{h\to 0}\frac{\sqrt[4]{16+h}-2}{h}$

(c) $\lim_{x\to 5}\frac{2^x-32}{x-5}$ (d) $\lim_{x\to\pi/4}\frac{\tan x-1}{x-\pi/4}$

(e) $\lim_{h\to 0}\frac{\cos(\pi+h)+1}{h}$ (f) $\lim_{t\to 1}\frac{t^4+t-2}{t-1}$

21. Find an equation of the tangent line to the curve $y=x^3-3x+1$ at the point

(2, 3).

22. Find an equation of the tangent line to the curve $y=\frac{xSin(x)}{1+x}$ at the point

(0, 0).

23. Find the derivative of the function $f(x)=3x^2-4x+1$ using the definition of the derivative at the point x = a.

24. Find the derivative of the function $f(x)=\sqrt{1-2x}$ using the definition of the derivative at the point x = a.

25. Let f be a differentiable function at x. Calculate the following limits in terms of $f'(x)$:

(a) $\lim_{h\to 0}\frac{f(x+h)-f(x)}{2h}$ (b) $\lim_{h\to 0}\frac{f(x-h)-f(x)}{h}$

(c) $\lim_{h\to 0}\frac{f(x+h^2)-f(x)}{h}$ (d) $\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}$

26. Use exercise 23 to compute the following limts:

(a) $\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{2h}$ (b) $\lim_{h\to 0}\frac{e^{-h}-1}{h}$

(c) $\lim_{h\to 0}\frac{\ln(1+h^2)}{h}$ (d) $\lim_{h\to 0}\frac{Cos(x+h)-Cos(x-h)}{h}$

27. Suppose that $f(x)$ and $g(x)$ are differentiable functions at the points x = a and x = b respectively. Furthermore, suppose that $f(a) = g(b)$.

(a) Express the limit $\lim_{h\to 0}\frac{f(a+h)-g(b+h)}{h}$ in terms of $f'(a)$ and $g'(b)$.

(b) Use part (a) to compute the limit $\lim_{h\to 0}\frac{\sqrt{1+h}-e^h}{h}$.

28. Suppose f is a function that satisfies the equation

$$f(x+y)=f(x)+f(y)+x^2y+xy^2$$

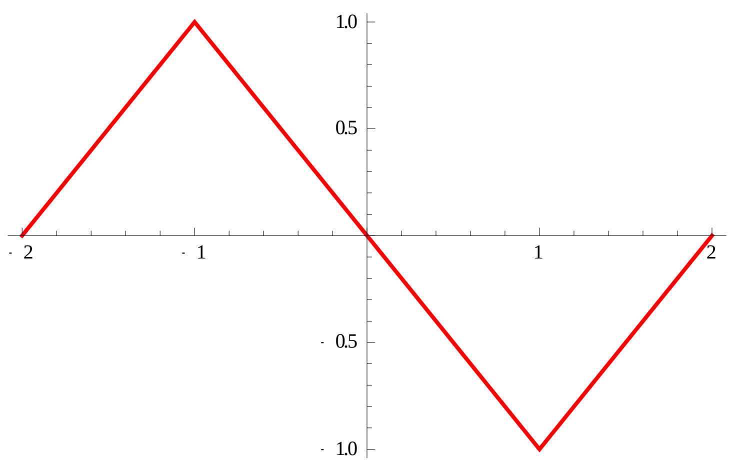
for all real numbers x and y. Suppose also that

$$\lim_{x\to 0}\frac{f(x)}{x}=1$$

Find:

(a) $f(0)$ (b) $f'(0)$ (c) $f'(x)$

29. The graph of the function $y=f(x)$ is displayed below



Draw the graph of $y=f'(x)$.

30. Differentiate

(a) $f(x) = (2x-3)^4(x^2+x+1)^5$

(b) $f(x) = \sqrt{\dfrac{x^2+1}{x^2+4}}$

(c) $f(x) = \cos(x^2)$

(d) $f(x) = \dfrac{1}{(1+\sec x)^2}$

(e) $f(x) = 10^{1-x^2}$

(f) $f(x) = \sin(e^x) + e^{\sin x}$

(g) $f(x) = \sin(\sin(\sin \text{x}))$

(h) $f(x) = \sqrt{x+\sqrt{x+\sqrt{x}}}$

31. Find dy/dx by implicit differentiation.

(a) $x^3 + y^3 = 1$

(b) $2x^3 + x^2y - xy^3 = 2$

(b) $\cos(xy) = 1 + \sin y$

(c) $\sqrt{x+y} = 1 + x^2y^2$

(d) $\tan^{-1}(x^2y) = x + xy^2$

(e) $\tan(x-y) = \dfrac{y}{1+x^2}$

32. Establish the derivative formula for the function $y = \tan^{-1} x$ by using implicit differentiation.

33. Find the derivative for the function $y = x^x$. [Hint: Use logarithmic differentiation]

34. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420. Find an expression for the number of bacteria after t hours.

35. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 mi away from the station.

36. Use linear approximation to estimate the value of $\sqrt{99.8}$

37. Find the absolute maximum and absolute minimum values of f on the given interval

(a) $f(x) = 12 + 4x - x^2$, [0, 5]

(b) $f(x) = 2x^3 - 3x^2 - 12x + 1$, [-2, 3]

(c) $f(x) = \sqrt[3]{x}(8-x)$, [0, 8]

(d) $f(x) = 2\cos(x) + \sin(2x)$, [0, π/2]

38. A cylindrical can without a top is made to contain V cm^3 of liquid. Find the dimensions that will minimize the amount of metal used to make the can.

39. State and prove the Mean-Value-Theorem.

40. Use the Mean-Value Theorem to prove the inequality $|\sin a - \sin b| \le |a-b|$ for all a and b.

41. If $f(1)=10$ and $f'(x)\ge 2$ for 1 ≤ x ≤ 4, how small can $f(4)$ possibly be?

42. For each of the functions below (i) Find intervals of increase or decrease. (ii) Find the local maximum and minimum values. (iii) Find the intervals of concavity and the inflection points. (iv) Use the information from parts (i)-(iii) to sketch the graph.

(a) $f(x)=x^3-12x+2$ (b) $f(x)=x\sqrt{6-x}$

(c) $\ln(x^4+27)$ (d) $f(x)=2\cos x+\cos^2 x$

43. For each of the functions below (i) Find the vertical and horizontal asymptotes. (ii) Find intervals of increase or decrease. (iii) Find the local maximum and minimum values. (iv) Find the intervals of concavity and the inflection points. (v) Use the information from parts (i)-(iv) to sketch the graph.

(a) $f(x)=1+\dfrac{1}{x}-\dfrac{1}{x^2}$ (b) $f(x)=\dfrac{x^2-4}{x^2+4}$

(b) $f(x)=\sqrt{x^2+1}-x$ (d) $f(x)=\dfrac{e^x}{1-e^x}$

44. Given that $\lim_{x\to a} f(x)=0$, $\lim_{x\to a} g(x)=0$, $\lim_{x\to a} h(x)=1$

$\lim_{x\to a} p(x)=\infty$, $\lim_{x\to a} q(x)=\infty$

which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

(a) $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ (b) $\lim\limits_{x\to a}\dfrac{f(x)}{p(x)}$ (c) $\lim\limits_{x\to a}\dfrac{h(x)}{p(x)}$

(d) $\lim\limits_{x\to a}\dfrac{p(x)}{f(x)}$ (e) $\lim\limits_{x\to a}\dfrac{p(x)}{q(x)}$ (f) $\lim\limits_{x\to a}[f(x)p(x)]$

(g) $\lim\limits_{x\to a}[h(x)p(x)]$ (h) $\lim\limits_{x\to a}[p(x)q(x)]$ (i) $\lim\limits_{x\to a}[f(x)-p(x)]$

(j) $\lim\limits_{x\to a}[p(x)-q(x)]$ (k) $\lim\limits_{x\to a}[p(x)+q(x)]$ (l) $\lim\limits_{x\to a}[f(x)]^{g(x)}$

(m) $\lim\limits_{x\to a}[f(x)]^{p(x)}$ (n) $\lim\limits_{x\to a}[h(x)]^{p(x)}$ (o) $\lim\limits_{x\to a}[p(x)]^{f(x)}$

(p) $\lim\limits_{x\to a}[p(x)]^{q(x)}$ (q) $\lim\limits_{x\to a}\sqrt[q(x)]{p(x)}$

45. Give an example where

(a) $\lim_{x\to 5} f(x)$ and $\lim_{x\to 5} g(x)$ do not exist, but $\lim_{x\to 5}(f(x)+g(x))=7$

(b) $\lim_{x\to 5} f(x)$ and $\lim_{x\to 5} g(x)$ do not exist, but $\lim_{x\to 5}[f(x)\cdot g(x)]=7$.

46. Find the limit. Use l'Hospital's Rule or more elementary or efficient techniques when appropriate. If l'Hospital's Rule doesn't apply, explain why.

(a) $\lim_{x\to(\pi/2)^+} \frac{\cos x}{1-\sin x}$

(b) $\lim_{t\to 0} \frac{e^{2t}-1}{\sin t}$

(c) $\lim_{x\to 0} \frac{x^2}{1-\cos x}$

(d) $\lim_{x\to 0} \frac{x3^x}{3^x-1}$

(e) $\lim_{x\to 0} \frac{\cos mx-\cos nx}{x^2}$

(f) $\lim_{x\to\infty} \sqrt{x}e^{-x/2}$

(g) $\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}$

(h) $\lim_{x\to 0^+} \frac{x^x-1}{\ln x+x-1}$

(i) $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$

(j) $\lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}$

47. (a) Express the integral $\int_4^7 (x^2+\sqrt{1+2x})dx$ as the limit of a Riemann sum of left rectangles L_n. Do not evaluate.

(b) Determine a region whose area is equal to $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{n}\left(5+\frac{2k}{n}\right)^{10}$. Do not evaluate.

48. Use the definition of the integral as a Riemann sum to evaluate the integral

(a) $\int_2^5 (4-2x)dx$

(b) $\int_1^4 (x^2-4x+2)dx$

(c) $\int_{-2}^0 (x^2-x)dx$

(d) $\int_0^1 (x^3-3x^2)dx$

49. Suppose F(x) is the antiderivative of f(x). Explain why every other antiderivative of f(x) must be of the form F(x) + C, where C is any constant.

50. Prove the Fundamental Theorem of Calculus. Namely, prove that if $f:(\alpha,\beta)$ $\to$ R is continuous and $a\in(\alpha,\beta)$ is any point in the interval where f(x) is

defined, then $F(x)=\int_a^x f(t)dt$ is one of its antiderivatives. In particular, every continuous, real valued function has an antiderivative.

51. Use the Fundamental Theorem of Calculus to find the derivative of

(a) $f(x)=\int_0^{\tan x}\sqrt{t+\sqrt{t}}\,dt$ (b) $f(x)=\int_{1-2x}^{1+2x} t\sin t\,dt$

(c) $f(x)=\int_{\sqrt{x}}^{2x}\tan^{-1} t\,dt$ (d) $f(x)=\int_{\cos x}^{\sin x}\ln(1+2t)dt$

52. Evaluate the integral by any known method

(a) $\int_{\pi/6}^{\pi}\sin\theta\, d\theta$ (b) $\int_0^1 (u+2)(u-3)du$

(c) $\int_0^4 (4-t)\sqrt{t}\,dt$ (d) $\int_1^{18}\sqrt{\frac{3}{z}}dz$

(e) $\int_1^2 \frac{v^3+3v^6}{v^4}dv$ (f) $\int_{1/2}^{1/\sqrt{2}}\frac{4}{\sqrt{1-x^2}}dx$

(g) $\int_1^2 \frac{e^{1/x}}{x^2}dx$ (h) $\int_{-\pi/3}^{\pi/2} x^3\sin x dx$

(i) $\int_0^1 \frac{e^z+1}{e^z+z}dz$ (j) $\int_1^2 x\sqrt{x-1}dx$

53. Use the properties of integrals to verify the inequality $\int_0^1\sqrt{1+x^2}\,dx \le \int_0^1\sqrt{1+x}dx$.

54. Calculate the following limits:

(a) $\lim\limits_{n\to\infty}\frac{1}{n}\left(\sqrt{\frac{1}{n}}+\sqrt{\frac{2}{n}}+\sqrt{\frac{3}{n}}+...+\sqrt{\frac{n}{n}}\right)$

(b) $\lim_{n\to\infty}\frac{\pi}{2n}\left(\sin\left(\frac{\pi}{2n}\right)+\sin\left(2\frac{\pi}{2n}\right)+\sin\left(3\frac{\pi}{2n}\right)+\cdots+\sin\left(n\frac{\pi}{2n}\right)\right)$